

# On a sequence of monogenic polynomials satisfying the Appell condition whose first term is a non-constant function

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## Abstract

In this paper we aim at constructing a sequence  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  of  $\mathbb{R}_{0,m}$ -valued polynomials which are monogenic in  $\mathbb{R}^{m+1}$  satisfying the Appell condition (i.e. the hypercomplex derivative of each polynomial in the sequence equals, up to a multiplicative constant, its preceding term) but whose first term  $\mathbf{M}_0^k(x) = \mathbf{P}_k(\underline{x})$  is a  $\mathbb{R}_{0,m}$ -valued homogeneous monogenic polynomial in  $\mathbb{R}^m$  of degree  $k$  and not a constant like in the classical case. The connection of this sequence with the so-called Fueter's theorem will also be discussed.

*Keywords:* Clifford algebras, monogenic functions, Appell sequences, Cauchy-Kovalevskaya extension technique, Fueter's theorem.

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## 1 Preliminaries

Let  $\mathbb{R}_{0,m}$  be the real Clifford algebra generated by the orthonormal basis  $\{e_1, \dots, e_m\}$  of the Euclidean space  $\mathbb{R}^m$  (see [6]). The multiplication in  $\mathbb{R}_{0,m}$  is associative and is determined by the relations:

$$\begin{aligned} e_j^2 &= -1, & j &= 1, \dots, m, \\ e_j e_k + e_k e_j &= 0, & 1 \leq j \neq k \leq m. \end{aligned}$$

A basis for  $\mathbb{R}_{0,m}$  is given by  $\{e_A : A \subset \{1, \dots, m\}\}$  where  $e_A = e_{j_1} \dots e_{j_k}$  with  $A = \{j_1, \dots, j_k\}$  and  $1 \leq j_1 < \dots < j_k \leq m$ . For the empty set  $\emptyset$ , we

put  $e_\emptyset = 1$ , the latter being the identity element. A general element  $a \in \mathbb{R}_{0,m}$  may thus be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

and its conjugate  $\bar{a}$  is given by

$$\bar{a} = \sum_A a_A \bar{e}_A, \quad \bar{e}_A = \bar{e}_{j_k} \dots \bar{e}_{j_1}, \quad \bar{e}_j = -e_j, \quad j = 1, \dots, m.$$

Observe that  $\mathbb{R}^{m+1}$  may be naturally embedded in the Clifford algebra  $\mathbb{R}_{0,m}$  by associating to any element  $x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$  the *paravector*  $x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_j e_j$ .

Let us recall that an  $\mathbb{R}_{0,m}$ -valued function  $f$  defined and continuously differentiable in an open set  $\Omega$  of  $\mathbb{R}^{m+1}$ , is said to be (left) monogenic in  $\Omega$  if and only if  $\partial_x f(x) = 0$  in  $\Omega$ , where

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}}$$

is the generalized Cauchy-Riemann operator in  $\mathbb{R}^{m+1}$  and

$$\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$$

is the Dirac operator in  $\mathbb{R}^m$ . Similarly, the same name is used for  $\mathbb{R}_{0,m}$ -valued functions defined in open subsets of  $\mathbb{R}^m$  which are null-solutions of the Dirac operator  $\partial_{\underline{x}}$ . Note that  $\partial_x$  factorizes the Laplacian, i.e.

$$\Delta_x = \sum_{j=0}^m \partial_{x_j}^2 = \partial_x \bar{\partial}_x = \bar{\partial}_x \partial_x,$$

and therefore every monogenic function is also harmonic. The monogenic functions are a fundamental object of study in Clifford analysis; and may be considered as a natural generalization to higher dimensions of the holomorphic functions of one complex variable (see e.g. [4, 7, 13]).

The hypercomplex derivative of a monogenic function  $f$  is defined as  $\frac{1}{2} \bar{\partial}_x f$  (see [12, 18]). As a monogenic function  $f$  clearly satisfies

$$\partial_{x_0} f = -\partial_{\underline{x}} f,$$

it easily follows that

$$\frac{1}{2} \bar{\partial}_x f = \partial_{x_0} f = -\partial_{\underline{x}} f.$$

Let us recall that a sequence of polynomials in which the index of each polynomial equals its degree is called a polynomial sequence. A polynomial sequence  $\{p_n(z)\}_{n \geq 0}$  is said to be an Appell sequence if it satisfies

$$p'_n(z) = np_{n-1}(z), \quad n \geq 1,$$

and  $p_0(z)$  is a non-zero constant (see [1]). Apart from the trivial example  $\{z^n\}_{n \geq 0}$ , there are important sequences in Mathematics which are Appell sequences for example the Bernoulli polynomials, the Hermite polynomials, and the Euler polynomials. Recently, this concept has been generalized to the Clifford analysis setting in [8, 9, 19] (see also [2, 3, 5, 11, 15]) as follows. A sequence  $\{P_n(x)\}_{n \geq 0}$  of  $\mathbb{R}_{0,m}$ -valued polynomials forms an Appell sequence if

- (i)  $\{P_n(x)\}_{n \geq 0}$  is a polynomial sequence;
- (ii) each  $P_n(x)$  is monogenic in  $\mathbb{R}^{m+1}$ , i.e.  $\partial_x P_n(x) = 0$  in  $\mathbb{R}^{m+1}$ ;
- (iii)  $\frac{1}{2} \bar{\partial}_x P_n(x) = nP_{n-1}(x)$ ,  $n \geq 1$ .

In [8, 9, 19] an important example of an Appell sequence of monogenic polynomials  $\{P_n^m(x)\}_{n \geq 0}$  with  $P_0^m(x) = 1$  was constructed, in which each term was of the form

$$P_n^m(x) = \sum_{j=0}^n \binom{n}{j} C_n(j) x_0^j \underline{x}^{n-j} = \sum_{j=0}^n T_j^n(x_0 + \underline{x})^{n-j} (x_0 - \underline{x})^j, \quad n \geq 0, \quad (1)$$

for suitable real numbers  $C_n(j)$ ,  $T_j^n$  (see [8, 9, 19]). The importance of these monogenic polynomials lies in the fact that they may be seen as the higher dimensional counterpart of the complex monomials  $z^n$ . At this point we must remark that that natural powers  $(x_0 + \underline{x})^n$  of the paravector variable are not monogenic for  $m \geq 2$ .

Note that the requirement of  $\{P_n(x)\}_{n \geq 0}$  being a polynomial sequence implies that the first term  $P_0(x)$  must be a constant. It is natural to ask whether we can consider sequences of monogenic polynomials satisfying the Appell condition (iii) but in which the first term is not a constant. In other words, we wish to drop condition (i) and keep conditions (ii)-(iii). The main goal of this note is to construct a sequence  $\{M_n^k(x)\}_{n \geq 0}$  of  $\mathbb{R}_{0,m}$ -valued polynomials which are monogenic in  $\mathbb{R}^{m+1}$  satisfying

$$\frac{1}{2} \bar{\partial}_x M_n^k(x) = nM_{n-1}^k(x), \quad n \geq 1, \quad (2)$$

with  $M_0^k(x) = \mathbf{P}_k(\underline{x})$  being a given but arbitrary  $\mathbb{R}_{0,m}$ -valued homogeneous polynomial of degree  $k$  which is monogenic in  $\mathbb{R}^m$ .

We believe that the sequence  $\{M_n^k(x)\}_{n \geq 0}$  provides a remarkable generalization of the well-known Appell sequences and may be of interest for those working in the subject. As far as we know, the methods we use to construct this sequence have never been employed in this framework. They allow us to arrive at our main result in a simple and elegant way.

In a forthcoming paper, we plan to consider even more general initial functions  $M_0^k(x)$  such as  $\mathbb{R}_{0,m}$ -valued homogeneous polynomials which are monogenic in  $\mathbb{R}^{m+1}$ .

## 2 Construction of the sequence $\{M_n^k(x)\}_{n \geq 0}$

Before starting with the construction of the sequence  $\{M_n^k(x)\}_{n \geq 0}$ , we shall first introduce some essential tools.

For a differentiable  $\mathbb{R}$ -valued function  $\phi$  and a differentiable  $\mathbb{R}_{0,m}$ -valued function  $g$ , we have

$$\partial_{\underline{x}}(\phi g) = \partial_{\underline{x}}(\phi)g + \phi(\partial_{\underline{x}}g). \quad (3)$$

Moreover, for a differentiable vector-valued function  $\underline{f} = \sum_{j=1}^m f_j e_j$ , we also have

$$\partial_{\underline{x}}(\underline{f}g) = (\partial_{\underline{x}}\underline{f})g - \underline{f}(\partial_{\underline{x}}g) - 2 \sum_{j=1}^m f_j (\partial_{x_j}g). \quad (4)$$

Throughout this paper we denote by  $\mathbf{P}_k(\underline{x})$  an  $\mathbb{R}_{0,m}$ -valued homogeneous polynomial of degree  $k \in \mathbb{N}_0$  which moreover is monogenic in  $\mathbb{R}^m$ , i.e.

$$\begin{aligned} \mathbf{P}_k(\underline{x}) &\in \mathbb{R}_{0,m}, \quad \partial_{\underline{x}}\mathbf{P}_k(\underline{x}) = 0, \quad \underline{x} \in \mathbb{R}^m, \\ \mathbf{P}_k(t\underline{x}) &= t^k \mathbf{P}_k(\underline{x}), \quad \underline{x} \in \mathbb{R}^m, \quad t \in \mathbb{R}. \end{aligned}$$

Let

$$\beta_k(n) = \begin{cases} n, & \text{if } n \text{ even} \\ 2k + m + n - 1, & \text{if } n \text{ odd} \end{cases}$$

for  $n \geq 1$  and put  $\beta_k(0) = 1$ . Using the Leibniz rules (3)-(4) as well as Euler's theorem for homogeneous functions, we can deduce the useful equality:

$$\partial_{\underline{x}}(\underline{x}^n \mathbf{P}_k(\underline{x})) = -\beta_k(n) \underline{x}^{n-1} \mathbf{P}_k(\underline{x}), \quad n \geq 1. \quad (5)$$

One basic result in Clifford analysis is the so-called Cauchy-Kovalevskaya extension technique (see [4, 7]), which we will make heavy use in our paper.

**Theorem 1** Every  $\mathbb{R}_{0,m}$ -valued function  $g(\underline{x})$  analytic in  $\mathbb{R}^m$  has a unique monogenic extension  $\text{CK}[g]$  to  $\mathbb{R}^{m+1}$ , which is given by

$$\text{CK}[g(\underline{x})](x) = \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \partial_{\underline{x}}^j g(\underline{x}). \quad (6)$$

Observe that a monogenic function  $f(x)$  can be reconstructed by knowing its restriction to  $\mathbb{R}^m$  using previous formula, i.e.

$$f(x) = \text{CK}[f(x)|_{\underline{x}_0=0}](x).$$

It is also worth noting that

$$\frac{1}{2} \bar{\partial}_x \text{CK}[g(\underline{x})](x) = -\partial_{\underline{x}} \text{CK}[g(\underline{x})](x) = \text{CK}[-\partial_{\underline{x}} g(\underline{x})](x). \quad (7)$$

We are now ready to construct our sequence  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  of  $\mathbb{R}_{0,m}$ -valued monogenic polynomials in  $\mathbb{R}^{m+1}$  which satisfies the Appell condition (2) and whose first term is  $\mathbf{M}_0^k(x) = \mathbf{P}_k(\underline{x})$ .

It is easy to check that we can put

$$\mathbf{M}_1^k(x) = \left( x_0 + \frac{\underline{x}}{2k+m} \right) \mathbf{P}_k(\underline{x}), \quad \mathbf{M}_2^k(x) = \left( x_0^2 + \frac{2x_0 \underline{x}}{2k+m} + \frac{\underline{x}^2}{2k+m} \right) \mathbf{P}_k(\underline{x})$$

as the next two elements in our sequence. Indeed, they are monogenic in  $\mathbb{R}^{m+1}$  and satisfy (2). Thus, it seems that we can conjecture that each term  $\mathbf{M}_n^k(x)$  in our sequence  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  will be of the form

$$\mathbf{M}_n^k(x) = H_n(x_0, \underline{x}) \mathbf{P}_k(\underline{x}), \quad n \geq 0,$$

where  $H_n(x_0, \underline{x})$  is a homogeneous polynomial of degree  $n$  with real coefficients in the two variables  $x_0$  and  $\underline{x}$ . Therefore

$$\mathbf{M}_n^k(x)|_{\underline{x}_0=0} = c_n \underline{x}^n \mathbf{P}_k(\underline{x}), \quad n \geq 0,$$

for some real constant  $c_n$  ( $n \geq 0$ ) with  $c_0 = 1$ . By Theorem 1, we have that

$$\mathbf{M}_n^k(x) = \text{CK}[\mathbf{M}_n^k(x)|_{\underline{x}_0=0}](x) = c_n \text{CK}[\underline{x}^n \mathbf{P}_k(\underline{x})](x), \quad n \geq 0.$$

Since  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  satisfy (2), it follows from (7) that

$$c_n \text{CK}[-\partial_{\underline{x}}(\underline{x}^n \mathbf{P}_k(\underline{x}))](x) = n c_{n-1} \text{CK}[\underline{x}^{n-1} \mathbf{P}_k(\underline{x})](x), \quad n \geq 1.$$

Then, identity (5) implies that

$$c_n = \frac{n c_{n-1}}{\beta_k(n)}, \quad n \geq 1.$$

This recurrence relation is easy to solve. Indeed,

$$c_n = \frac{n!}{\prod_{s=0}^n \beta_k(s)}, \quad n \geq 0.$$

We thus get

$$\mathbf{M}_n^k(x) = \frac{n!}{\prod_{s=0}^n \beta_k(s)} \text{CK}[\underline{x}^n \mathbf{P}_k(\underline{x})](x), \quad n \geq 0. \quad (8)$$

Moreover, using formula (6) and identity (5), we obtain

$$\mathbf{M}_n^k(x) = \left( \sum_{j=0}^n \binom{n}{j} C_{k,n}(j) x_0^j \underline{x}^{n-j} \right) \mathbf{P}_k(\underline{x}), \quad n \geq 0, \quad (9)$$

$$\text{with } C_{k,n}(j) = \frac{(n-j)!}{\prod_{s=0}^{n-j} \beta_k(s)}.$$

**Proposition 1** *Suppose that  $\mathbf{M}_0^k(x) = \mathbf{P}_k(\underline{x})$  is an  $\mathbb{R}_{0,m}$ -valued homogeneous polynomial of degree  $k$  which is monogenic in  $\mathbb{R}^m$ . Then  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  given by (9) is a sequence of  $\mathbb{R}_{0,m}$ -valued polynomials which are monogenic in  $\mathbb{R}^{m+1}$  and satisfies the Appell condition (2).*

**Remark:** *For the particular case  $k = 0$ ,  $\mathbf{P}_k(\underline{x}) = 1$ ,  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  equals the Appell sequence  $\{\mathbf{P}_n^m(x)\}_{n \geq 0}$  given by (1).*

It should be noticed that  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  is not a polynomial sequence. Indeed,  $\mathbf{M}_n^k(x)$  ( $n \geq 0$ ) is a  $\mathbb{R}_{0,m}$ -valued homogeneous polynomial of degree  $k + n$  which is monogenic in  $\mathbb{R}^{m+1}$ .

It is worth pointing out that  $\mathbf{M}_n^k(x)$  can be written as

$$\mathbf{M}_n^k(x) = \left( A_n(x_0, r) + \frac{\underline{x}}{r} B_n(x_0, r) \right) \mathbf{P}_k(\underline{x}), \quad r = |\underline{x}|, \quad n \geq 0, \quad (10)$$

where  $A_n$  and  $B_n$  are  $\mathbb{R}$ -valued continuously differentiable functions in  $\mathbb{R}^2$ . The monogenicity of  $\mathbf{M}_n^k(x)$  implies that  $A_n$  and  $B_n$  satisfy the Vekua-type system (see [30])

$$\begin{cases} \partial_{x_0} A_n - \partial_r B_n &= \frac{2k + m - 1}{r} B_n \\ \partial_{x_0} B_n + \partial_r A_n &= 0. \end{cases}$$

Monogenic functions of the form (10) are called axial monogenic of degree  $k$  and are an important class of functions in Clifford analysis (see [17, 26, 27]). Thus, we can also assert that  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  is a sequence of axial monogenic functions of degree  $k$  which satisfies the Appell condition (2).

### 3 Connection with Fueter's theorem

Fueter's theorem constitutes a method for generating axial monogenic functions starting from holomorphic functions in the upper half of the complex plane. First proved by R. Fueter (see [10]) in the setting of quaternionic analysis, this result was later extended to Clifford analysis in [25, 23, 28]. For other references on this subject we refer the reader to e.g. [14, 16, 20, 21, 22, 24, 29].

For odd dimensions  $m$ , Fueter's theorem runs as follows (see e.g. [28]).

**Theorem 2** *Let  $f(z) = u(x, y) + iv(x, y)$  ( $z = x + iy$ ) be a holomorphic function in some open subset  $\Xi \subset \{z \in \mathbb{C} : y > 0\}$ . Put  $\underline{\omega} = \underline{x}/r$ , with  $r = |\underline{x}|$ . If  $m$  is odd, then the function*

$$\text{Ft}[f(z), \mathbf{P}_k(\underline{x})](x) = \Delta_x^{k + \frac{m-1}{2}} [(u(x_0, r) + \underline{\omega} v(x_0, r)) \mathbf{P}_k(\underline{x})]$$

*is monogenic in  $\tilde{\Omega} = \{x \in \mathbb{R}^{m+1} : (x_0, r) \in \Xi\}$ .*

In [11] it was shown that for  $m$  being odd the Appell sequence  $\{\mathbf{P}_n^m(x)\}_{n \geq 0}$  given by (1) and Fueter's theorem applied to the complex monomials  $z^n$  are closely related. More precisely,  $\mathbf{P}_n^m(x)$  equals (up to a multiplicative constant)  $\text{Ft}[z^{n+m-1}, 1](x)$  for  $n \geq 0$ . Since for  $k = 0$  and  $\mathbf{P}_k(\underline{x}) = 1$ ,  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  coincides with  $\{\mathbf{P}_n^m(x)\}_{n \geq 0}$ , we may expect that a similar connection should exist between our sequence  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  and Fueter's theorem. In fact, it was computed in [20] (among other examples) that for odd dimensions  $m$

$$\begin{aligned} \text{Ft}[z^n, \mathbf{P}_k(\underline{x})](x) &= (-1)^{k + \frac{m-1}{2}} (2k + m - 1)!! \alpha_k(n) \\ &\quad \times \text{CK}[\underline{x}^{n-(2k+m-1)} \mathbf{P}_k(\underline{x})](x), \quad n \geq 2k + m - 1, \end{aligned} \quad (11)$$

with

$$\alpha_k(n) = \begin{cases} \frac{n!!}{(n-2k-m+1)!!}, & \text{if } n \text{ even} \\ \alpha_k(n-1), & \text{if } n \text{ odd,} \end{cases}$$

and where  $(\cdot)!!$  denotes the double factorial. We also note that

$$\text{Ft} [z^n, \mathbf{P}_k(\underline{x})] (x) = 0, \quad \text{for } n < 2k + m - 1.$$

Thus, using (8) and (11), we get:

**Proposition 2** *For odd dimensions  $m$ , each term  $\mathbf{M}_n^k(x)$  of the sequence  $\{\mathbf{M}_n^k(x)\}_{n \geq 0}$  equals (up to a multiplicative constant)  $\text{Ft} [z^{n+2k+m-1}, \mathbf{P}_k(\underline{x})] (x)$ .*

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